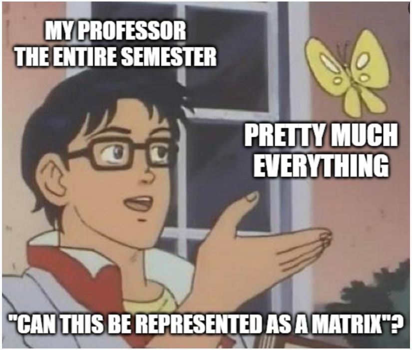
Games, graphs, and machines



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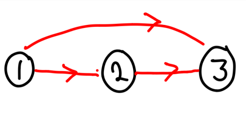
Warm up

Find the adjacency matrix A and its powers A^2, A^3, A^4, $\cdots$ for the following graph.



$A^0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

↳ Length 0 paths

$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Length 1 paths

$$A^2 = \begin{array}{c} \begin{array}{ccc} & & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Length 2 paths

$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Length 3 paths

Why does A^k count length k paths?

Theorem
The (i,j) entry if A^k is the number of paths from vertex i to vertex j.

Suppose $n = 3$. True for $K=1$.

$$A^2_{i,j} = \overbrace{\underbrace{\underbrace{A_{i,1}}_{\text{Length 1 from } i \text{ to } 1} \cdot \underbrace{A_{1,j}}_{\text{Length 1 from } 1 \text{ to } j}}_{\text{Length 2 from } i \text{ to } j \text{ whose penult stop is } 1} + \underbrace{A_{i,2} \cdot A_{2,j}}_{\text{Length 2 from } i \text{ to } j \text{ whose penult stop is } 2} + \underbrace{A_{i,3} \cdot A_{3,j}}_{\text{— '' — stop is } 3}}^{\text{all len 2 from } i \text{ to } j}$$

$A_{i,1}$ A_{i2} A_{i3}

A_{1j}
A_{2j}
A_{3j}

Why does A^k count length k paths?

$$A^3_{i,j} = \boxed{A^2_{i,1} \cdot A_{1,j}} + \boxed{A^2_{i,2} \cdot A_{2,j}} + \boxed{A^2_{i,3} \cdot A_{3,j}}$$

Len 1 fom 1 to j

Len 2 fom i to 1

Len 3 fom i to j
whose penult
stop is 1

Len 3 from
i to j
whose penult
stop is 2

....stop
is 3

All len 3 from i to j

Why does A^k count length k paths?

$$A^4_{i,j} = A^3_{i,1} \cdot A_{1,j} + A^3_{i,2} \cdot A_{2,j} + A^3_{i,3} \cdot A_{3,j}$$

A^5

A^6

$\vdots$

A^K

$\vdots$

"Induction"

Truth for K

$\Downarrow$

Truth for $K+1$

$+$ Truth for $K=1$

$\Rightarrow$ Trth for $K = 1, 2, 3, \ldots$

Sum of powers

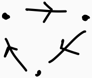
What do the entries of $A + A^2 + A^3 + A^4$ represent?

Paths of length 1 or 2 or 3 or 4.

We say that G is *acyclic* if it has no (directed) cycle. Suppose G is acyclic and has 100 vertices. What can you say about A^{100}?

Acyclic $\Rightarrow$ no self loops

Cycle = path that begins & ends at the same vertex.



Path of length 100

cycle!

$$V_0 \to V_1 \to V_2 \to \cdots \to V_{100}$$

Longest path

Let G be a graph with adjacency matrix A. Using A, how will you find the longest possible path in G?

Length of longest path

$= $ Largest n s.t.

$A^n \neq 0$.